

ON NON-CONGRUENT NUMBERS WITH 1 MODULO 4 PRIME FACTORS

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ABSTRACT. In this paper, we use the 2-decent method to find a series of odd non-congruent numbers $\equiv 1 \pmod{8}$ whose prime factors are $\equiv 1 \pmod{4}$ such that the congruent elliptic curves have second lowest Selmer groups, which includes Li and Tian's result [5] as special cases.

1. INTRODUCTION

The congruent number problem is about when a positive integer can be the area of a rational right triangle. A positive integer n is a non-congruent number is equivalent to that the congruent elliptic curve

$$E := E^{(n)} : y^2 = x^3 - n^2x \quad (1)$$

has Mordell-Weil rank zero. In [3] and [4], Feng obtained several series of non-congruent numbers for $E^{(n)}$ with the lowest Selmer groups. In [5], Li and Tian obtained a series of non-congruent numbers whose prime factors are $\equiv 1 \pmod{8}$ such that $E^{(n)}$ has second lowest Selmer groups. The essential tool of the above results is the 2-descend method of elliptic curves. In this paper, we will use this method to get a series of odd non-congruent numbers whose prime factors are $\equiv 1 \pmod{4}$ such that $E^{(n)}$ has second lowest Selmer groups, which includes Li and Tian's result as special cases.

Suppose n is a square-free integer such that $n = p_1 \cdots p_k \equiv 1 \pmod{8}$ and primes $p_i \equiv 1 \pmod{4}$, then by quadratic reciprocity law $\left(\frac{p_i}{p_j}\right) = \left(\frac{p_j}{p_i}\right)$.

Definition 1.1. Suppose $n = p_1 \cdots p_k \equiv 1 \pmod{8}$ and $p_i \equiv 1 \pmod{4}$. The graph $G(n) := (V, A)$ associated to n is a simple undirected graph with vertex set $V := \{\text{prime } p \mid n\}$ and edge set $A := \{\overline{pq} : \left(\frac{p}{q}\right) = -1\}$.

Recall for a simple undirected graph $G = (V, A)$, a partition $V = V_0 \cup V_1$ is called *even* if for any $v \in V_i$ ($i = 0, 1$), $\#\{v \rightarrow V_{1-i}\}$ is even. G is called an *odd graph* if the only even partition is the trivial partition $V = \emptyset \cup V$. Then our main result is:

Theorem 1.2. *Suppose $n = p_1 \cdots p_k \equiv 1 \pmod{8}$ and $p_i \equiv 1 \pmod{4}$. If the graph $G(n)$ is odd and $\delta(n)$ (as given by (16)) is 1, then for the congruent elliptic curve $E = E^{(n)}$,*

$$\text{rank}_{\mathbb{Z}}(E(\mathbb{Q})) = 0 \text{ and } \text{III}(E/\mathbb{Q})[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

As a consequence, n is a non-congruent number.

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The following Corollary is Li and Tian's result, cf. [5]:

Corollary 1.3. *Suppose $n = p_1 \cdots p_k$ and $p_i \equiv 1 \pmod{8}$. If the graph $G(n)$ is odd and the Jacobi symbol $\left(\frac{1+\sqrt{-1}}{n}\right) = -1$, then for $E = E^{(n)}$,*

$$\text{rank}_{\mathbb{Z}}(E(\mathbb{Q})) = 0 \text{ and } \text{III}(E/\mathbb{Q})[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

As a consequence, n is a non-congruent number.

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2. REVIEW OF 2-DESCENT METHOD.

In this section, we recall the 2-descent method of computing the Selmer groups of elliptic curves. This section follows [5] pp 232-233, also cf. [1] §5 and [7] X.4.

For an isogeny $\varphi : E \rightarrow E'$ of elliptic curves defined over a number field K , one has the following fundamental exact sequence

$$0 \rightarrow E'(K)/\varphi E(K) \rightarrow S^{(\varphi)}(E/K) \rightarrow \text{III}(E/K)[\varphi] \rightarrow 0. \quad (2)$$

Moreover, if $\psi : E' \rightarrow E$ is another isogeny, for the composition $\psi \circ \varphi : E \rightarrow E$, then the following diagram of exact sequences commutes (cf. [8] p 5):

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \vdots \downarrow \iota_1 & & \vdots \downarrow \iota_2 & & \downarrow \\ 0 & \longrightarrow & E'(K)/\varphi E(K) & \longrightarrow & S^{(\varphi)}(E/K) & \longrightarrow & \text{III}(E/K)[\varphi] \longrightarrow 0 \\ & & \downarrow \psi & & \downarrow & & \downarrow \\ 0 & \longrightarrow & E(K)/\psi\varphi E(K) & \longrightarrow & S^{(\psi\varphi)}(E/K) & \longrightarrow & \text{III}(E/K)[\psi\varphi] \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & E(K)/\psi E'(K) & \longrightarrow & S^{(\psi)}(E'/K) & \longrightarrow & \text{III}(E'/K)[\psi] \longrightarrow 0 \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

Now suppose n is a fixed odd positive square-free integer, $K = \mathbb{Q}$, and E/\mathbb{Q} , E'/\mathbb{Q} , φ , $\psi = \varphi^\vee$ are given by

$$E = E^{(n)} : y^2 = x^3 - n^2x, \quad E' = \widehat{E^{(n)}} : y^2 = x^3 + 4n^2x,$$

$$\varphi : E \rightarrow E', (x, y) \mapsto \left(\frac{y^2}{x^2}, \frac{y(x^2 + n^2)}{x^2}\right),$$

$$\psi : E' \rightarrow E, (x, y) \mapsto \left(\frac{y^2}{4x^2}, \frac{y(x^2 - 4n^2)}{8x^2}\right).$$

Then $\varphi\psi = [2], \psi\varphi = [2]$. In this case ι_1 and ι_2 are exact. Let $\tilde{S}^{(\psi)}(E'/\mathbb{Q})$ denote the image of $S^{(\psi\varphi)}(E/\mathbb{Q})$ in $S^{(\psi)}(E'/\mathbb{Q})$. Then

$$\#\text{III}(E/\mathbb{Q})[\varphi] = \frac{\#S^{(\varphi)}(E/\mathbb{Q})}{\#E'(\mathbb{Q})/\varphi E(\mathbb{Q})}, \quad \#\text{III}(E'/\mathbb{Q})[\psi] = \frac{\#S^{(\psi)}(E'/\mathbb{Q})}{\#E(\mathbb{Q})/\psi E'(\mathbb{Q})},$$

and

$$\#\text{III}(E/\mathbb{Q})[2] = \frac{\#S^{(\varphi)}(E/\mathbb{Q}) \cdot \#\tilde{S}^{(\psi)}(E'/\mathbb{Q})}{\#E'(\mathbb{Q})/\varphi E(\mathbb{Q}) \cdot \#E(\mathbb{Q})/\psi E'(\mathbb{Q})}. \quad (3)$$

Similarly,

$$\#\text{III}(E'/\mathbb{Q})[2] = \frac{\#S^{(\psi)}(E'/\mathbb{Q}) \cdot \#\tilde{S}^{(\varphi)}(E/\mathbb{Q})}{\#E(\mathbb{Q})/\psi E'(\mathbb{Q}) \cdot \#E'(\mathbb{Q})/\varphi E(\mathbb{Q})}. \quad (4)$$

The 2-descent method to compute the Selmer groups $S^{(\varphi)}(E/\mathbb{Q})$ and $S^{(\psi)}(E'/\mathbb{Q})$ is as follows (cf. [7] for general elliptic curves). Let

$$S = \{\text{prime factors of } 2n\} \cup \{\infty\},$$

$$\mathbb{Q}(S, 2) = \{b \in \mathbb{Q}^\times / \mathbb{Q}^{\times 2} : 2 \mid \text{ord}_p(b), \forall p \notin S\}.$$

Note that $\mathbb{Q}(S, 2)$ is represented by factors of $2n$ and we identify these two sets. By the exact sequence

$$0 \rightarrow E'(\mathbb{Q})/\varphi E(\mathbb{Q}) \xrightarrow{i} \mathbb{Q}(S, 2) \xrightarrow{j} WC(E/\mathbb{Q})[\varphi],$$

where

$$i : (x, y) \mapsto x, \quad O \mapsto 1, \quad (0, 0) \mapsto 4n^2, \quad j : d \mapsto \{C_d/\mathbb{Q}\}$$

and C_d/\mathbb{Q} is the homogeneous space for E/\mathbb{Q} defined by the equation

$$C_d : dw^2 = d^2 + 4n^2 z^4, \quad (5)$$

the φ -Selmer group $S^{(\varphi)}(E/\mathbb{Q})$ is then

$$S^{(\varphi)}(E/\mathbb{Q}) \cong \{d \in \mathbb{Q}(S, 2) : C_d(\mathbb{Q}_p) \neq \emptyset, \forall p \in S\}. \quad (6)$$

Similarly, suppose

$$C'_d : dw^2 = d^2 - n^2 z^4. \quad (7)$$

The ψ -Selmer group $S^{(\psi)}(E'/\mathbb{Q})$ is then

$$S^{(\psi)}(E'/\mathbb{Q}) \cong \{d \in \mathbb{Q}(S, 2) : C'_d(\mathbb{Q}_p) \neq \emptyset, \forall p \in S\}. \quad (8)$$

The method to compute $\tilde{S}^{(\varphi)}(E/\mathbb{Q})$ follows from [1] §5, Lemma 10:

Lemma 2.1. *Let $d \in S^{(\varphi)}(E/\mathbb{Q})$. Suppose (σ, τ, μ) is a nonzero integer solution of $d\sigma^2 = d^2\tau^2 + 4n^2\mu^2$. Let \mathcal{M}_b be the curve corresponding to $b \in \mathbb{Q}^\times / \mathbb{Q}^{\times 2}$ given by*

$$\mathcal{M}_b : dw^2 = d^2 t^4 + 4n^2 z^4, \quad d\sigma w - d^2 \tau t^2 - 4n^2 \mu z^2 = bu^2. \quad (9)$$

Then $d \in \tilde{S}^{(\varphi)}(E/\mathbb{Q})$ if and only if there exists $b \in \mathbb{Q}(S, 2)$ such that \mathcal{M}_b is locally solvable everywhere.

Note that the existence of σ, τ, μ follows from Hasse-Minkowski theorem (cf. [6]).

3. LOCAL COMPUTATION

We need a modification of the Legendre symbol. For $x \in \mathbb{Q}_p$ or $x \in \mathbb{Q}$ such that $\text{ord}_p(x)$ is even, we set

$$\left(\frac{x}{p}\right) := \left(\frac{xp^{-\text{ord}_p(x)}}{p}\right). \quad (10)$$

Thus $(\frac{\cdot}{p})$ defines a homomorphism from $\{x \in \mathbb{Q}^\times / \mathbb{Q}^{\times 2} : \text{ord}_p(x) \text{ is even}\}$ to $\{\pm 1\}$.

3.1. Computation of Selmer groups. In this subsection, we will find the conditions when C_d or C'_d is locally solvable. We will not give details since one only need to consider the valuations and quadratic residue.

Lemma 3.1. $d \in S^{(\varphi)}(E/\mathbb{Q})$ if and only if d satisfies

- (1) $d > 0$ has no prime factor $p \equiv 3 \pmod{4}$;
- (2) $\left(\frac{n/d}{p}\right) = 1$ for all odd $p \mid d$;
- (3) $\left(\frac{d}{p}\right) = 1$ for all odd $p \mid (2n/d)$;
- (4) if $2 \mid d$, $n \equiv \pm 1 \pmod{8}$.

Proof. In this case $C_d : dw^2 = d^2t^4 + 4n^2z^4$. It is obvious that $C_d(\mathbb{R}) \neq \emptyset \Leftrightarrow d > 0$. Assume $d > 0$.

- (i) If $2 \nmid d \mid n$, then $C_d : w^2 = d(t^4 + 4(n/d)^2z^4)$.
 - $p = 2$. $C_d(\mathbb{Q}_2) \neq \emptyset \Leftrightarrow d \equiv 1 \pmod{4}$.
 - $p \mid d$. $C_d(\mathbb{Q}_p) \neq \emptyset \Leftrightarrow \left(\frac{n/d}{p}\right) = 1$ and $p \equiv 1 \pmod{4}$.
 - $p \nmid d$. $C_d(\mathbb{Q}_p) \neq \emptyset \Leftrightarrow \left(\frac{d}{p}\right) = 1$.
- (ii) If $2 \mid d \mid 2n$, then $C_d : w^2 = d(t^4 + (2n/d)^2z^4)$.
 - $p = 2$. $C_d(\mathbb{Q}_2) \neq \emptyset \Leftrightarrow d \equiv 2 \pmod{8}$, $n \equiv \pm 1 \pmod{8}$.
 - $2 \neq p \mid d$. $C_d(\mathbb{Q}_p) \neq \emptyset \Leftrightarrow \left(\frac{n/d}{p}\right) = 1$ and $p \equiv 1 \pmod{4}$.
 - $p \nmid d$. $C_d(\mathbb{Q}_p) \neq \emptyset \Leftrightarrow \left(\frac{d}{p}\right) = 1$.

Combining (i) and (ii) follows the lemma. \square

Lemma 3.2. $d \in S^{(\psi)}(E'/\mathbb{Q})$ if and only if d satisfies

- (1) $d \equiv \pm 1 \pmod{8}$ or $n/d \equiv \pm 1 \pmod{8}$
- (2) $\left(\frac{n/d}{p}\right) = 1$ for all $p \mid d, p \equiv 1 \pmod{4}$;
- (3) $\left(\frac{d}{p}\right) = 1$ for all $p \mid (n/d), p \equiv 1 \pmod{4}$.

Proof. In the case $C'_d : dw^2 = d^2t^4 - n^2z^4$.

- (i) If $2 \mid d$, consider the 2-valuation of each side, we see $C'_d(\mathbb{Q}_2) = \emptyset$.
- (ii) If $2 \nmid d \mid n$, then $C'_d : w^2 = d(t^4 - (n/d)^2z^4)$.
 - $p = 2$. $C'_d(\mathbb{Q}_2) \neq \emptyset \Leftrightarrow d \equiv \pm 1 \pmod{8}$ or $n/d \equiv \pm 1 \pmod{8}$.
 - $p \mid d$. $C'_d(\mathbb{Q}_p) \neq \emptyset \Leftrightarrow \left(\frac{n/d}{p}\right) = 1$ or $\left(\frac{-n/d}{p}\right) = 1$.
 - $p \nmid d$. $C'_d(\mathbb{Q}_p) \neq \emptyset \Leftrightarrow \left(\frac{d}{p}\right) = 1$ or $\left(\frac{-d}{p}\right) = 1$.

Combining (i) and (ii) follows the lemma. \square

3.2. Computation of the images of Selmer groups. Suppose $0 < 2d \in S^{(\varphi)}(E/\mathbb{Q})$, d is odd with no $\equiv 3 \pmod{4}$ prime factor, we want to find a necessary condition for $2d \in \tilde{S}^{(\varphi)}(E/\mathbb{Q})$. Write $2d = \tau^2 + \mu^2$ and select the triple (σ, τ, μ) in Lemma 2.1 to be $(2n, n\tau/d, \mu)$. Then the defining equations of \mathcal{M}_{4ndb} in (9) can be written as

$$w^2 = 2d(t^4 + (n/d)^2 z^4), \quad w - \tau t^2 - (n/d)\mu z^2 = bu^2. \quad (11)$$

By abuse of notations, we denote the above curve by \mathcal{M}_b . We use the notation $O(p^m)$ to denote a number with p -adic valuation $\geq m$.

The case $p \mid d$. For $i_p \equiv \tau/\mu \pmod{p\mathbb{Z}_p}$, $i_p \in \mathbb{Z}_p$ and $i_p^2 = -1$, then

$$p \mid (\tau - i_p \mu), \quad p \nmid (\tau + i_p \mu).$$

It's easy to see $v(t) = v(z)$, we may assume that $z = 1$, $t^2 \equiv \pm \frac{i_p n}{d} \pmod{p}$, then \mathcal{M}_b is given by

$$\mathcal{M}_b : w^2 = 2d(t^4 + (n/d)^2), \quad w - \tau t^2 - (n/d)\mu = bu^2.$$

(i) If $v(bu^2) = m \geq 3$, then by $w^2 = (\tau t^2 + \frac{n\mu}{d} + O(p^m))^2 = 2d(t^4 + \frac{n^2}{d^2})$,

$$\left(\mu t^2 - \frac{n\tau}{d}\right)^2 = O(p^m).$$

Let $t^2 = \frac{n\tau}{d\mu} + \beta$, where $v(\beta) = \alpha \geq \frac{m}{2}$, then

$$\begin{aligned} w^2 &= 2d \left(\left(\frac{n}{d}\right)^2 + \left(\frac{n\tau}{d\mu}\right)^2 + 2\frac{n\tau}{d\mu}\beta + \beta^2 \right) \\ &= \frac{4n^2}{\mu^2} \left(1 + \frac{\tau\mu}{n}\beta + \frac{d\mu^2}{2n^2}\beta^2 \right), \end{aligned}$$

Take the square root on both sides, then

$$\begin{aligned} w &= \pm \frac{2n}{\mu} \left(1 + \frac{1}{2} \left(\frac{\tau\mu}{n}\beta + \frac{d\mu^2}{2n^2}\beta^2 \right) - \frac{1}{8} \left(\frac{\tau\mu}{n}\beta \right)^2 + O(p^{3\alpha-3}) \right) \\ &= \pm \left(\frac{2n}{\mu} + \tau\beta + n\mu \left(\frac{\mu\beta}{2n} \right)^2 + O(p^{3\alpha-2}) \right), \end{aligned}$$

but on the other hand,

$$w = \tau t^2 + \frac{n\mu}{d} + bu^2 = \frac{2n}{\mu} + \tau\beta + bu^2.$$

The sign must be positive and

$$bu^2 = n\mu \left(\frac{\mu\beta}{2n} \right)^2 + O(p^{3\alpha-2}),$$

thus $p \mid b$, $\left(\frac{b/p}{p}\right) = \left(\frac{n\mu/p}{p}\right)$, $\left(\frac{n/b}{p}\right) = \left(\frac{\mu}{p}\right) = \left(\frac{2\tau}{p}\right)$.

(ii) If $v(bu^2) = m \leq 2$ and $t^2 \equiv \frac{i_p n}{d} \pmod{p}$, let $t^2 = \frac{i_p n}{d} + p\alpha i_p$, then

$$w^2 = 2d \cdot p\alpha i_p \cdot \left(\frac{2i_p n}{d} + p\alpha i_p \right) = -4p^2 \cdot \frac{n\alpha}{p} \left(1 + \frac{pd\alpha}{2n} \right),$$

and

$$\begin{aligned}
w_1 &= \frac{w}{p} = \pm 2i_p \sqrt{\frac{n\alpha}{p}} \left(1 + \frac{pd\alpha}{4n} + O(p^2) \right), \\
bu^2 &= w - \tau t^2 - \frac{n\mu}{d} \\
&= \pm 2pi_p \sqrt{\frac{n\alpha}{p}} \left(1 + \frac{pd\alpha}{4n} \right) - \frac{i_p \tau n}{d} - \frac{n\mu}{d} - \tau \alpha i_p p + O(p^3) \\
&= -\frac{p^2 i_p \tau}{n} \left(\sqrt{\frac{n\alpha}{p}} \mp \frac{n}{p\tau} \right)^2 - \frac{ni_p}{2d\tau} (\tau - i_p \mu)^2 \pm 2p^2 i_p \sqrt{\frac{n\alpha}{p}} \frac{d\alpha}{4n} + O(p^3).
\end{aligned}$$

If $v(bu^2) = 2$, then $\sqrt{\frac{n\alpha}{p}} \equiv \pm \frac{n}{p\tau} \pmod{p}$, and

$$\begin{aligned}
bu^2 &= -\frac{ni_p}{2d\tau} (\tau - i_p \mu)^2 \pm 2p^2 i_p \sqrt{\frac{n\alpha}{p}} \frac{d\alpha}{4n} + O(p^3) \\
&= \frac{-ni_p (\tau - i_p \mu)^3 (3\tau + i_p \mu)}{8d\tau^3} + O(p^3) \\
&= \frac{-ni_p (\tau - i_p \mu)^3}{2d\tau^2} + O(p^3) = O(p^3),
\end{aligned}$$

which is impossible! Thus $v(bu^2) = 1$ and $p \mid b$,

$$\left(\frac{b/p}{p} \right) = \left(\frac{-pi_p \tau / n}{p} \right) = \left(\frac{2p\tau/n}{p} \right), \text{ or } \left(\frac{n/b}{p} \right) = \left(\frac{2\tau}{p} \right).$$

(iii) If $v(bu^2) = m \leq 2$ and $t^2 \equiv -i_p(n/d) \pmod{p}$, then

$$\begin{aligned}
bu^2 &= w - \tau t^2 - (n/d)\mu = (\tau i_p - \mu)n/d + O(p) \\
&= 2i_p \tau n/d + O(p) = (1 + i_p)^2 \cdot \frac{n}{d} \cdot \tau + O(p),
\end{aligned}$$

thus $p \nmid b$ and $\left(\frac{b}{p} \right) = \left(\frac{\tau}{p} \right) \left(\frac{n/d}{p} \right)$.

Note that $2\tau \equiv \tau + \mu i_p \pmod{p}$ and $\left(\frac{2n/d}{p} \right) = 1$, hence we have

Lemma 3.3. *The curve \mathcal{M}_b defined by (11) is locally solvable at $p \mid d$ if and only if*

$$\text{either } p \mid b, \left(\frac{n/b}{p} \right) = \left(\frac{\tau + \mu i_p}{p} \right); \quad \text{or } p \nmid b, \left(\frac{b}{p} \right) = \left(\frac{\tau + \mu i_p}{p} \right).$$

The case $p \mid \frac{n}{d}$. In this case t is a p -adic unit if and only if w is so.

(i) If $v(w) = v(t) = 0$, then $w \equiv \pm \sqrt{2d}t^2 \pmod{p}$ and $(\pm \sqrt{2d} - \tau)t^2 \equiv bu^2 \pmod{p}$. Since $(\sqrt{2d} - \tau)(\sqrt{2d} + \tau) = 2d - \tau^2 = \mu^2$ and $\sqrt{2d} \pm \tau$ are co-prime, $\text{ord}_p(\sqrt{2d} - \tau)$ is even and $\left(\frac{\sqrt{2d} - \tau}{p} \right)$ is well defined. Then \mathcal{M}_b is locally solvable if and only if

$$p \nmid b, \left(\frac{2d}{p} \right) = 1 \text{ and } \left(\frac{b}{p} \right) = \left(\frac{\sqrt{2d} - \tau}{p} \right).$$

(ii) If $v(z) = 0$ and $w = pw_1, t = pt_1$, then $w_1^2 = 2d(p^2 t_1^2 + (\frac{n}{pb})^2 z^4)$, $w_1 \equiv \pm \sqrt{2d} \frac{n}{pd} z^2 \pmod{p}$ and $bu^2/p \equiv (\pm \sqrt{2d} - \mu) \frac{n}{pd} z^2 \pmod{p}$. Thus \mathcal{M}_b is locally

solvable if and only if

$$p \mid b, \left(\frac{2d}{p}\right) = 1 \text{ and } \left(\frac{n/(db)}{p}\right) = \left(\frac{\sqrt{2d} - \mu}{p}\right).$$

Note that

$$2(\sqrt{2d} - \tau)(\sqrt{2d} - \mu) = (\tau + \mu - \sqrt{2d})^2 \Rightarrow \left(\frac{\sqrt{2d} - \mu}{p}\right) = \left(\frac{2(\sqrt{2d} - \tau)}{p}\right).$$

From now on, suppose $n = p_1 \cdots p_k \equiv 1 \pmod{8}$ and $p_i \equiv 1 \pmod{4}$. Pick $i_p \in \mathbb{Z}_p$ such that $i_p^2 = -1$, then

$$\sqrt{2d} - \tau = -(\tau + \mu i_p) \cdot \frac{1}{2} \left(1 - \frac{\sqrt{2d}}{\tau + \mu i_p}\right)^2.$$

Note that $\left(\frac{2d}{p}\right) = 1$, we have

Lemma 3.4. \mathcal{M}_b defined by (11) is locally solvable at $p \mid \frac{n}{d}$ if and only if

$$\begin{aligned} p \mid b, \quad \left(\frac{2d}{p}\right) = 1 \text{ and } \left(\frac{n/b}{p}\right) &= \left(\frac{\tau + \mu i_p}{p}\right) \left(\frac{2}{p}\right), \\ \text{or } p \nmid b, \quad \left(\frac{2d}{p}\right) = 1 \text{ and } \left(\frac{b}{p}\right) &= \left(\frac{\tau + \mu i_p}{p}\right) \left(\frac{2}{p}\right). \end{aligned}$$

By Lemmas 2.1, 3.1, 3.3 and 3.4, and we have

Proposition 3.5. Suppose $n = p_1 \cdots p_k \equiv 1 \pmod{8}$ and $p_i \equiv 1 \pmod{4}$, then $2d \in S^{(\varphi)}(E/\mathbb{Q})$ if and only if $d > 0$ and $\left(\frac{2n/d}{p}\right) = 1$ for $p \mid d$, $\left(\frac{2d}{p}\right) = 1$ for $p \mid \frac{n}{d}$.

In this case $2d \in \tilde{S}^{(\varphi)}(E/\mathbb{Q})$ only if there exists $b \in \mathbb{Q}(S, 2)$ satisfying:

(1) If $p \mid d$, $i_p \equiv \tau/\mu \pmod{p\mathbb{Z}_p}$, $i_p^2 = -1$,

$$p \mid b, \quad \left(\frac{n/b}{p}\right) = \left(\frac{\tau + \mu i_p}{p}\right), \quad \text{or } p \nmid b, \quad \left(\frac{b}{p}\right) = \left(\frac{\tau + \mu i_p}{p}\right).$$

(2) If $p \mid \frac{n}{d}$, $i_p^2 = -1$,

$$p \mid b, \quad \left(\frac{n/b}{p}\right) = \left(\frac{2(\tau + \mu i_p)}{p}\right), \quad \text{or } p \nmid b, \quad \left(\frac{b}{p}\right) = \left(\frac{2(\tau + \mu i_p)}{p}\right).$$

4. PROOF OF THE MAIN RESULT

4.1. Some facts about graph theory. We now recall some notations and results in graph theory, cf. [3, 4].

Definition 4.1. Let $G = (V, A)$ be a simple undirected graph. Suppose $\#V = k$. The *adjacency matrix* $M(G) = (a_{ij})$ of G is the $k \times k$ matrix defined as

$$a_{ij} := \begin{cases} 0, & \text{if } \overline{v_i v_j} \notin A; \\ 1, & \text{if } \overline{v_i v_j} \in A. \end{cases} \quad (12)$$

The *Laplace matrix* $L(G)$ of G is defined as

$$L(G) = \text{diag}\{d_1, \dots, d_k\} - M(G) \quad (13)$$

where d_i is the degree of v_i .

Theorem 4.2. *Let G be a simple undirected graph and $L(G)$ its Laplace matrix.*

- (1) *The number of even partitions of V is 2^{k-1-r} , where $r = \text{rank}_{\mathbb{F}_2} L(G)$.*
- (2) *The graph G is odd if and only if $r = k - 1$.*
- (3) *If G is odd, then the equations*

$$L(G) \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix}$$

has solutions if and only if $t_1 + \cdots + t_k = 0$.

Proof. The proof of the first two parts follows from [3]. We have a bijection

$$\begin{aligned} \mathbb{F}_2^k / \{(0, \dots, 0), (1, \dots, 1)\} &\xrightarrow{\sim} \{\text{partitions of } V\} \\ (c_1, \dots, c_k) &\longmapsto (V_0, V_1) \end{aligned}$$

where $V_i = \{v_j : c_j = i \ (1 \leq j \leq k)\}$, $i \in \{0, 1\}$.

Regard $L(G) = \text{diag}\{d_1, \dots, d_k\} - (a_{ij})$ as a matrix over \mathbb{F}_2 . If

$$L(G) \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix} \in \mathbb{F}_2^k,$$

then if $v_i \in V_t$, $t \in \{0, 1\}$,

$$\begin{aligned} b_i &= d_i c_i + \sum_{j=1}^k a_{ij} c_j = \sum_{j=1}^k a_{ij} (c_i + c_j) \\ &= \sum_{j=1}^k a_{ij} (t + c_j) = \sum_{c_j=1-t} a_{ij} = \#\{v_i \rightarrow V_{1-t}\} \in \mathbb{F}_2. \end{aligned}$$

- (1) The number of even partitions is

$$\frac{1}{2} \# \left\{ (c_1, \dots, c_k) \in \mathbb{F}_2^k : L(G) \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\} = 2^{k-1-r}.$$

- (2) follows from (1) easily.

(3) Since L is of rank $k-1$, the image space of L is of dimensional $k-1$, but it lies in the hyperplane $x_1 + \cdots + x_k = 0$, thus they coincide and the result follows. \square

4.2. Graph $G(n)$ and Selmer groups of E and E' . From now on, we suppose

$$n = p_1 \cdots p_k \equiv 1 \pmod{8} \text{ and } p_i \equiv 1 \pmod{4}.$$

Recall for an integer a prime to n , the Jacobi symbol $\left(\frac{a}{n}\right) = \prod_{p|n} \left(\frac{a}{p}\right)$, which is extended to a multiplicative homomorphism from $\{a \in \mathbb{Q}^\times / \mathbb{Q}^{\times 2} : \text{ord}_p(a) \text{ even for } p \mid n\}$ to $\{\pm 1\}$. Set

$$\left[\frac{a}{n}\right] := \frac{1}{2} \left(1 - \left(\frac{a}{n}\right)\right). \quad (14)$$

The symbol $\left[\frac{\cdot}{n}\right]$ is an additive homomorphism from $\{a \in \mathbb{Q}^\times / \mathbb{Q}^{\times 2} : \text{ord}_p(a) \text{ even for } p \mid n\}$ to \mathbb{F}_2 .

By definition, the adjacency matrix $M(G(n))$ has entries $a_{ij} = \left[\frac{p_i}{p_j}\right]$. For $0 < d \mid n$, we denote by $\{d, \frac{n}{d}\}$ the partition $\{p : p \mid d\} \cup \{p : p \mid \frac{n}{d}\}$ of $G(n)$.

The following proposition is a translation of results in Lemma 3.1 and Lemma 3.2:

Proposition 4.3. *Given a factor d of n .*

- (1) *For the Selmer group $S^{(\varphi)}(E/\mathbb{Q})$,*

(1-a) $d \in S^{(\varphi)}(E/\mathbb{Q})$ if and only if $d > 0$ and $\{d, n/d\}$ is an even partition of $G(n)$;

(1-b) Suppose

$$c_i = \begin{cases} 1, & \text{if } p_i \mid d, \\ 0, & \text{if } p_i \mid \frac{n}{d}; \end{cases} \quad t_i = \left\lfloor \frac{2}{p_i} \right\rfloor.$$

Then $2d \in S^{(\varphi)}(E/\mathbb{Q})$ if and only if $d > 0$ and

$$L(G) \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix}.$$

(2) For the Selmer group $S^{(\psi)}(E'/\mathbb{Q})$,

(2-a) $d \in S^{(\psi)}(E'/\mathbb{Q})$ if and only if $d \equiv \pm 1 \pmod{8}$ and $\{d, n/d\}$ is an even partition of $G(n)$;

(2-b) $2d \notin S^{(\psi)}(E'/\mathbb{Q})$.

Proof. One only has to show (1-b), the rest is easy. For any i , let $[i]$ be the set of j such that p_i and p_j are both prime divisors of d or n/d . Then

$$d_i c_i + \sum_{j \neq i} a_{ij} c_j = \sum_{j \neq i} a_{ij} (c_i + c_j) = \sum_{j \notin [i]} a_{ij} = \left\lfloor \frac{d}{p_i} \right\rfloor \text{ or } \left\lfloor \frac{n/d}{p_i} \right\rfloor.$$

Then (1-b) follows from Lemma 3.1. \square

Applying Theorem 4.2(3) to Proposition 4.3, then we have

Corollary 4.4. *If $G(n)$ is odd, there exists a unique factor $0 < d < \sqrt{2n}$ of n such that*

$$S^{(\varphi)}(E/\mathbb{Q}) = \{1, 2d, 2n/d, n\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

and

$$S^{(\psi)}(E'/\mathbb{Q}) = \{\pm 1, \pm n\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

For the d given in Corollary 4.4, write $2d = \tau^2 + \mu^2$. If $2d \in \tilde{S}^{(\varphi)}(E/\mathbb{Q})$, we suppose b satisfies the condition that \mathcal{M}_b defined by (11) is locally solvable everywhere. Suppose $c' = (c'_1, \dots, c'_k)^T$ and $t' = (t'_1, \dots, t'_k)^T$ are given by

$$c'_j = \begin{cases} 1, & \text{if } p_j \mid b, \\ 0, & \text{if } p_j \nmid b; \end{cases} \quad t'_j = \begin{cases} \left\lfloor \frac{\tau + \mu i_{p_j}}{p_j} \right\rfloor, & \text{if } p_j \mid d, \\ \left\lfloor \frac{2(\tau + \mu i_{p_j})}{p_j} \right\rfloor, & \text{if } p_j \mid \frac{n}{d}. \end{cases}$$

By Proposition 3.5, $Lc' = t'$, i.e., $Lv = t'$ has a solution $v = c'$, which means that the summation of t'_j must be zero in \mathbb{F}_2 by Theorem 4.2(3).

Definition 4.5. Suppose n is given such that $G(n)$ is an odd graph. For the unique factor d given in Corollary 4.4, write $2d = \tau^2 + \mu^2$ and $\frac{2n}{d} = \tau'^2 + \mu'^2$, Let $i \in \mathbb{Z}/n\mathbb{Z}$ be defined by

$$i \equiv \frac{\tau}{\mu} \pmod{d}, \quad i \equiv \frac{\tau'}{\mu'} \pmod{\frac{n}{d}}. \quad (15)$$

We define

$$\delta(n) := \left\lfloor \frac{\tau + \mu i}{n} \right\rfloor + \left\lfloor \frac{2}{d} \right\rfloor \in \mathbb{F}_2. \quad (16)$$

Then the following is a consequence of Proposition 3.5.

Corollary 4.6. *If $G(n)$ is odd and $\delta(n) = 1$, then*

$$\tilde{S}^{(\varphi)}(E/\mathbb{Q}) = \{1\}.$$

Proof. Let λ^* be the \mathbb{F}_2 -rank of $\tilde{S}^{(\varphi)}(E/\mathbb{Q})$, λ be the \mathbb{F}_2 -rank of $S^{(\varphi)}(E/\mathbb{Q})$, then $\lambda = 2$. The existence of the Cassels' skew-symmetric bilinear form on III implies that the difference $\lambda - \lambda^*$ is even.

By the above analysis, $\delta(n) = \sum_j t'_j \neq 0$, thus $2d \notin \tilde{S}^{(\varphi)}(E/\mathbb{Q})$, we have $\lambda^* < \lambda$, $\lambda^* = 0$. \square

Remark. If we replace d by $\frac{n}{d}$ in the definition, $\delta(n)$ is invariant. Indeed, $[\frac{2}{d}] = [\frac{-2}{n/d}]$. For the other term,

$$\left[\frac{\tau + \mu i}{n} \right] = \left[\frac{\tau + \mu i}{d} \right] + \left[\frac{\tau + \mu i'}{n/d} \right]$$

where $i \equiv \tau/\mu \pmod{d}$, $i' \equiv \tau'/\mu' \pmod{n/d}$. Let $u = (\tau\tau' - \mu\mu')/2$, $v = (\tau\mu' - \mu\tau')/2$, then

$$\begin{aligned} u + vi &= (\tau + \mu i)(\tau' + \mu' i)/2 \equiv \tau(\tau' + \mu' \cdot \frac{\tau}{\mu}) \\ &\equiv \tau\mu(\tau'\mu + \tau\mu')/\mu^2 \equiv (\tau + \mu)^2/\mu^2 \cdot v/2 \pmod{d}. \end{aligned}$$

Similarly, $u + vi' \equiv (\tau' + \mu')^2/\mu'^2 \cdot v/2 \pmod{n/d}$. If we interchange d and n/d , $\delta(n)$ will differ

$$\begin{aligned} &\left[\frac{\tau + \mu i}{d} \right] + \left[\frac{\tau + \mu i'}{n/d} \right] + \left[\frac{\tau' + \mu' i'}{n/d} \right] + \left[\frac{\tau' + \mu' i}{d} \right] \\ &= \left[\frac{2(u + vi)}{d} \right] + \left[\frac{2(u + vi')}{n/d} \right] = \left[\frac{v}{d} \right] + \left[\frac{v}{n/d} \right] \\ &= \left[\frac{v}{n} \right] = \left[\frac{n}{v} \right] = 0 \in \mathbb{F}_2. \end{aligned}$$

Thus $\delta(n)$ does not change, which implies that $\delta(n)$ does not depend on the choice of d, τ, μ and only depend on n .

4.3. Proof of the main result.

Proof of Theorem 1.2. We shall use the fundamental exact sequence (2) and the commutative diagram in §2 frequently.

Since $E(\mathbb{Q})_{\text{tor}} \cap \psi E'(\mathbb{Q}) = \{O\}$ and $\#E(\mathbb{Q})_{\text{tor}} = 4$, $\#E(\mathbb{Q})/\psi E'(\mathbb{Q}) \geq 4$. Since $G(n)$ is odd, $\#S^{(\psi)}(E'/\mathbb{Q}) = 4$ and $\#E(\mathbb{Q})/\psi E'(\mathbb{Q}) = 4$, by (2), $\text{III}(E'/\mathbb{Q})[\psi] = 0$. Apparently $\tilde{S}^{(\psi)}(E'/\mathbb{Q}) \supseteq E(\mathbb{Q})/\psi E'(\mathbb{Q})$ and thus $\#\tilde{S}^{(\psi)}(E'/\mathbb{Q}) = 4$.

By Corollary 4.6, $\tilde{S}^{(\varphi)}(E/\mathbb{Q}) = \{1\}$, then $\#E'(\mathbb{Q})/\varphi E(\mathbb{Q}) = 1$. The facts $\#E(\mathbb{Q})/\psi E'(\mathbb{Q}) = 4$ and $E(\mathbb{Q})_{\text{tor}} \cong (\mathbb{Z}/2\mathbb{Z})^2$ imply that $\#E(\mathbb{Q})/2E(\mathbb{Q}) = 4$ and

$$\text{rank}_{\mathbb{Z}} E(\mathbb{Q}) = \text{rank}_{\mathbb{Z}} E'(\mathbb{Q}) = 0.$$

From $\text{III}(E'/\mathbb{Q})[\psi] = E'(\mathbb{Q})/\varphi E(\mathbb{Q}) = 0$, the diagram tells us that

$$\text{III}(E/\mathbb{Q})[2] \cong \text{III}(E/\mathbb{Q})[\varphi] \cong S^{(\varphi)}(E/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

and (4) tells us that

$$\text{III}(E'/\mathbb{Q})[2] \cong \text{III}(E'/\mathbb{Q})[\psi] \cong 0.$$

Hence $\text{III}(E'/\mathbb{Q})[2^\infty] = 0$ and $\text{III}(E'/\mathbb{Q})[2^k\psi] = 0$. By the exact sequence

$$0 \rightarrow \text{III}(E/\mathbb{Q})[\varphi] \rightarrow \text{III}(E/\mathbb{Q})[2^k] \rightarrow \text{III}(E'/\mathbb{Q})[2^{k-1}\psi],$$

we have for every $k \in \mathbb{N}_+$,

$$\text{III}(E/\mathbb{Q})[2^k] \cong \text{III}(E/\mathbb{Q})[\varphi] \cong (\mathbb{Z}/2\mathbb{Z})^2,$$

and thus $\text{III}(E/\mathbb{Q})[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2$. \square

Proof of Corollary 1.3. In this case, $d = 1$ and $\tau = \mu = 1$, $\delta(n) = \left\lfloor \frac{1+\sqrt{-1}}{n} \right\rfloor$, thus the result follows. \square

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